B.Sc. part-I(H), paper-I 30-07-2020.

Q.1 Define Cosets and let $a \in G$ be arbitrary and H be a subgroup of a group G. Then $Ha = H = aH \Leftrightarrow a \in H$.

Soln: Cosets:→ Let H be a subgroup of a group (G, o). Let $a \in G$ be arbitrary. We define

$aH = \{ah : h \in H\}$ and $Ha = \{ha : h \in H\}$

then aH is called left coset of H in G generated by a, and Ha is called right coset of H in G generated by a.

proof: ~~Let $a \in G$ be arbitrary and let~~

Let H be a subgroup of G and let $a \in G$ be ~~arbit~~ arbitrary.

StepI. We have to show that

$Ha = H \Leftrightarrow a \in H$

Let us first suppose $Ha = H$, to show $a \in H$

$e \in H, a \in H \Rightarrow ea \in Ha \Rightarrow a \in Ha$

$\Rightarrow a \in H$. For $H = Ha$

Now, let $a \in H$, to show $Ha = H$

Let $xa \in Ha \Rightarrow x \in H$

$\Rightarrow x \in H, a \in H$; For $a \in H$

$\Rightarrow xa \in H$ [∵ H is a subgroup]

Thus, any $xa \in Ha \Rightarrow xa \in H$

This prove that $Ha \subset H$ ——— (1)

$a \in H \Rightarrow a^{-1} \in H$ [∵ H is a subgroup]

For any, $y \in H, a^{-1} \in H \Rightarrow ya^{-1} \in H$

$\Rightarrow (ya^{-1})a \in Ha$

$\Rightarrow y \in Ha$ for $ya^{-1}a = ye = y$

Thus any, $y \in H \Rightarrow y \in Ha$ ——— (2)

$\Rightarrow H \subset Ha$

Now from from 1 and (2), We get

$H = Ha$

Step II. We have to show that $aH = H \Leftrightarrow a \in H$

We can prove Step II by making the parallel arguments as in I.

Proved

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Q.2 If a and b are arbitrary distinct elements of a group G and H is any subgroup of G, then

$$Ha = Hb \Leftrightarrow ab^{-1} \in H$$
$$aH = bH \Leftrightarrow b^{-1}a \in H$$

Proof:- Let a and b be arbitrary elements of a group G such that $a \neq b$.
Let e be the identity of $G \Rightarrow e \in H$
Firstly, we shall show that

$$Ha = Hb \Leftrightarrow ab^{-1} \in H$$
$$Ha = Hb \Rightarrow (Ha)b^{-1} = (Hb)(b^{-1})$$
$$\Rightarrow H(ab^{-1}) = H(bb^{-1}) = He = H$$
$$\Rightarrow H(ab^{-1}) = H \Rightarrow ab^{-1} \in H$$

Conversely. $ab^{-1} \in H \Rightarrow H(ab^{-1}) = H$
$$\Rightarrow (Hab^{-1})(b) = Hb$$
$$\Rightarrow (Ha)(b^{-1}b) = Hb$$
$$\Rightarrow (Ha)e = Hb \Rightarrow Ha = Hb$$

Therefore, we have $Ha = Hb \Leftrightarrow ab^{-1} \in H$

Similarly; we can prove $aH = bH \Leftrightarrow b^{-1}a \in H$.

Proved.

Q.3. There exists a one-one correspondence between a subgroups H of G and a left coset aH of H in G.

Proof. Let us consider a map $f: H \to aH$, defined by

$$f(h) = ah \text{ for every } h \in H$$

The mapping f is clearly onto aH, because for each $ah \in aH$ then exist an $h \in H$ such that $f(h) = ah$
Now, Let $h_1, h_2 \in H$ such that $f(h_1) = f(h_2)$ i.e. $ah_1 = ah_2$

$$\Rightarrow h_1 = h_2$$
$$\Rightarrow f \text{ is one-one}$$

Thus, f maps H such that f is one-one and onto.
Hence, there is a one-to-one correspondence between H and aH.

Proved.